Towards A Categorical Approach of Transformational Music Theory

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Abstract:

Transformational music theory mainly deals with group and group actions on sets, which are usually constituted by chords. For example, neo-Riemannian theory uses the dihedral group D_{24} to study transformations between major and minor triads, the building blocks of classical and romantic harmony. Since the developments of neo-Riemannian theory, many developments and generalizations have been proposed, based on other sets of chords, other groups, etc. However music theory also face problems for example when defining transformations between chords of different cardinalities, or for transformations that are not necessarily invertible. This paper introduces a categorical construction of musical transformations based on category extensions using groupoids. This can be seen as a generalization of a previous work which aimed at building generalized neo-Riemannian groups of transformations based on group extensions. The categorical extension construction allows the definition of partial transformations between different set-classes. Moreover, it can be shown that the typical wreath products groups of transformations can be recovered from the category extensions by "packaging" operators and considering their composition.

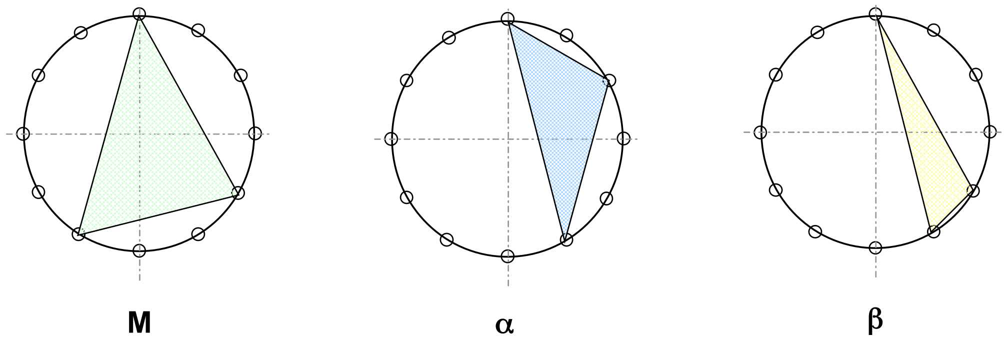
1 Introduction

After the pionneering work of David Lewin [1], the field of music theory has seen huge developments with regards to transformational models and their use for musical analysis. These theories have relied heavily on the group structure, in which group elements are seen as operations between elements. In neo-Riemannian theory, the classical set of elements was originally constituted by the major and minor chords, and the typical corresponding group of transformations is isomorphic to the dihedral group D_{24} of 24 elements, whether it acts through the famous L, R and P operations or through the transpositions and inversions operators [2, 3, 4, 5], or many others (see for example the Schritt-Wechsel group) [6].

Following its application to major/minor triads, generalizations have been actively researched. For example, transformational theory has also been applied to other sets of chords [7]. Different groups of transformations than the dihedral one have been proposed. Julian Hook's UTT group contains for example all transformations of triads respecting transposition, and has at its a core a wreath product construction [8, 9]. Wreath products were also studied by Robert Peck in a more general setting [10]. More recently, Robert Peck introduced imaginary transformations [11], in which quaternion groups, dicyclic groups and other extraspecial groups appear. A different approach has been undertaken by the author [12], in an attempt to unify all these different groups, in which generalized neo-Riemannian groups of musical transformations are built as extensions.

However, current transformational theories face multiple problems. One of the first is that transformational theory sometimes fail to provide interesting groups of transformations for some sets of chords (an example will be given below). The second one is that transformational theories have also failed to provide a solution to the cardinality problem, namely finding transformations between chords of different cardinalities. While Childs [13] studied neo-Riemannian theory applied to seventh chords, his model does not include triads. Hook [14] introduced another approach, namely cross-type transformations, to circumvent this problem.

In this paper, we introduce categorical constructions for musical transformations with the aim of generalizing existing constructions. This work can be viewed as a generalization of the previous work on group extensions, using groupoids instead of groups, and building groupoid extensions. Note that a categorical approach to music theory has been heavily investigated in the



(a) (b) (c)

Figure 1: Set classes M={0,4,7} (major chord) (a), α={0,2,5} (b) and β={0,4,5} (c)

book *The Topos of Music* by G. Mazzola [15]. Mazzola deplores in particular that *"Although the theory of categories has been around since the early 1940s and is even recognized by computer scientists, no attempt is visible in AST (Atonal Set Theory) to deal with morphisms between pcsets, for example"*. The first section highlights some of the limitations of current transformational theories based on particular examples. The second part introduces a categorical construction for musical transformations. Finally, the third part explores the relation between the categories constructed in section 2 and the more familiar groups of musical transformations, showing in particular how wreath products are naturally recovered from the category extensions.

2 On some limitations of transformational theories

2.1 Groups of transformations acting on three set-classes

Consider set classes {0,4,7}, {0,2,5}, {0,4,5}, which we label as M, α and β, as represented in Figure 1.

These set-classes have a well-defined root, and therefore transposition operators T_i can be defined for M, α and β, namely there is a simply transitive group action of $\mathbb{Z}_{12}$ on the 36-element set S of M, α and βchords.

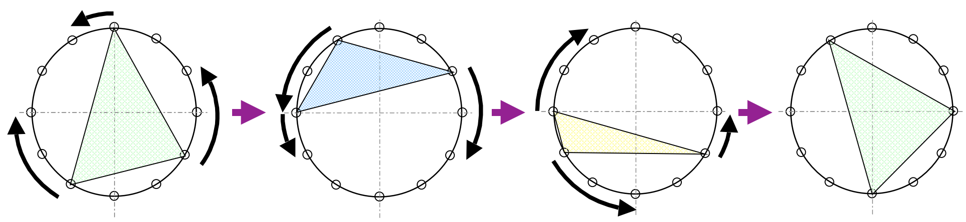


Figure 2: The action of the voice-leading VL operation on set-classes M, α and β

There also exists voice-leading transformations VL between these set-classes. For example, if one represents a chord as an ordered set (x, y, z), where x is the root, the VL transformation is defined as (all operations are understood mod 12):

$$VL : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} z+2 \\ x-1 \\ y-2 \end{pmatrix}$$

If one represents chords as n_t where n is the root and t the type of the chord, then this transformation is defined as :

$$VL : \begin{pmatrix} n_M \\ n_\alpha \\ n_\beta \end{pmatrix} \longmapsto \begin{pmatrix} n-3_\alpha \\ n-5_\beta \\ n-5_M \end{pmatrix}$$

Another voice-leading transformation VL' has a similar action, defined as :

$$VL' : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} z+4 \\ x+1 \\ y \end{pmatrix}$$

or equivalently :

$$VL' : \begin{pmatrix} n_M \\ n_\alpha \\ n_\beta \end{pmatrix} \longmapsto \begin{pmatrix} n+1_\alpha \\ n-3_\beta \\ n-3_M \end{pmatrix}$$

The VL and VL' operations are clearly contextual [16] since their action depends on the type of the chord on which they act. Notice moreover that $VL^{-3} = VL'^{21} = T_1$ and that $\langle T_1, VL\rangle = \langle T_1, VL'\rangle = \mathbb{Z}_{36}$

More generally, if one tries to apply the construction in [12] to build a group extension G of simply transitive musical transformations as :

$$1 \to \mathbb{Z}_{12} \to G \to \mathbb{Z}_3 \to 1$$

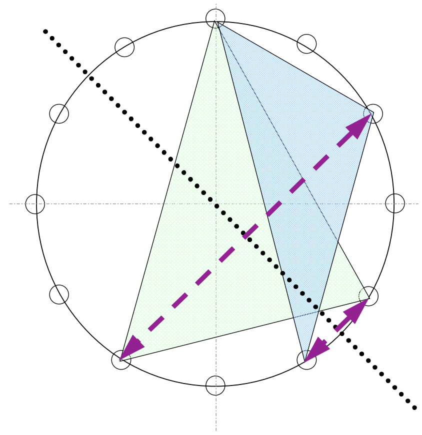
one ends up with only two abelian groups, namely $G = \mathbb{Z}_{12} \times \mathbb{Z}_3$ or $G = \mathbb{Z}_{36}$. The reason for this is that $\mathbb{Z}_{12}$ has too few automorphisms (remember that $Aut(\mathbb{Z}_{12}) = \mathbb{Z}_2 \times \mathbb{Z}_2$) and therefore there can be no action of $\mathbb{Z}_3$ on $\mathbb{Z}_{12}$ except for the trivial one. This means that group structures such as semidirect products, as is the case for the dihedral group D_{24} of neo-Riemannian theory, cannot exist for sets containing three different types of chords. Whereas $G = \mathbb{Z}_{12} \times \mathbb{Z}_3$ corresponds to the trivial direct product, i.e the trivial 2-cocycle, there also exists a non-trivial 2-cocycle which corresponds to $G = \mathbb{Z}_{36}$. As shown in [12], non-trivial 2-cocycles give rise to contextual group actions on chords, the VL and VL' operations being such examples. However, since the group is abelian (i.e the 2-cocycle is symmetric) the left- and right- actions of these transformations coincide.

An important part of the litterature about neo-Riemannian theory has focused on left- and right- actions and their interrelations [17, 18, 19], and in particular the commuting property between these actions (recall that left- and right-actions always commute in a group extension). In our case, the set S does not provide such richness. One could circumvent this problem by considering group extensions of the form :

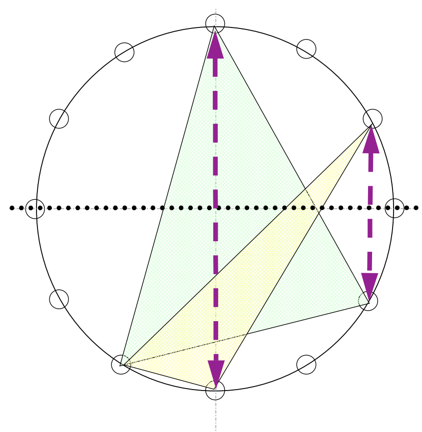
$$1 \to \mathbb{Z}_3 \to G \to \mathbb{Z}_{12} \to 1$$

but in this case, the transposition operator T_1 would not be well-defined anymore, since $\mathbb{Z}_{12}$ would no longer be a normal subgroup of G in the general case.

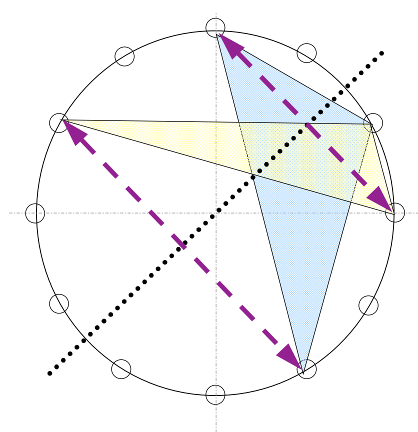
Moreover, the consideration of group extensions of the form $1 \to \mathbb{Z}_{12} \to G \to \mathbb{Z}_3 \to 1$ limits the contextual and/or voice-leading transformations that can be applied to chords of S, even when the 2-cocycle is non-trivial. Consider for example the following transformations :



(a)



(b)



(c)

Figure 3: (a), (b), (c) : three contextual operations acting on pairs of set-classes. No operation can be applied to all set classes altogether.

$$I_{M\leftrightarrow\alpha}:\begin{pmatrix}x\\y\\z\end{pmatrix}\longmapsto\begin{pmatrix}x\\(2x-3)-y\\(2x-3)-z\end{pmatrix},i.e\begin{pmatrix}n_M\\n_\alpha\end{pmatrix}\longmapsto\begin{pmatrix}n_\alpha\\n_M\end{pmatrix}$$

$$I_{M\leftrightarrow\beta}:\begin{pmatrix}x\\y\\z\end{pmatrix}\longmapsto\begin{pmatrix}(2z+4)-y\\(2z+4)-x\\z\end{pmatrix}i.e\begin{pmatrix}n_M\\n_\beta\end{pmatrix}\longmapsto\begin{pmatrix}n+2_\beta\\n-2_M\end{pmatrix}$$

$$I_{\alpha\leftrightarrow\beta}:\begin{pmatrix}x\\y\\z\end{pmatrix}\longmapsto\begin{pmatrix}(2y-1)-z\\y\\(2y-1)-x\end{pmatrix}i.e\begin{pmatrix}n_\alpha\\n_\beta\end{pmatrix}\longmapsto\begin{pmatrix}n-2_\beta\\n+2_\alpha\end{pmatrix}$$

These inversion-like transformations are represented in Figure 3. Each one of them is an involution, just as the L, R and P operations are. However, they can only be applied to the indicated pair of set-classes. In other terms, these operations are partial and cannot form a group of transformations since the closure condition would not be satisfied. We propose a way to unify these transformations in the next section.

2.2 Transformations between chords of different cardinalities

The work of Childs [13] has shown that neo-Riemannian constructions can be applied to seventh chords. In view of [12] and since seventh chords have a well-defined root, it is indeed possible to envision a group extension acting on seventh chords. However Childs' work does not include triads.

Since $Aut(\mathbb{Z}_{12}) = \mathbb{Z}_2 \times \mathbb{Z}_2$, one could consider that there is enough room for transformations of a set of four set-classes and their transpositions. However Hook [14] (see footnote 9 p.5) has argued against putting all set-classes in a single set on which transformations could be applied because (we paraphrase):

1. A transformation may not have the same meaning as its inverse, especially for transformations between different set-classes.

2. Transformations should be well-defined on the whole set of chords (this is the totality requirement for groups).

3. Some transformations may not have inverses at all.

4. Different sets of chords may not have the same cardinality and defining transformations between them would be problematic if not impossible. (Note that we differentiate between the cardinality of a set of chords, i.e the number of chords of the same set-class that constitutes the set, and the cardinality of a set-class or chord, i.e the number of pitch-classes that constitutes it.

We have seen in the previous part examples of transformations which do not apply on the whole set of M, α and β chords, i.e totality is lost. On another level, Cohn's model of triadic progression involves the transformation between major or minor chords, in each case a set of 12 elements, to augmented triads, a set of 4 elements. In that case these transformations are surjective and therefore have no formal inverse, although in Cohn's model one can freely choose the major/minor image of a given augmented triad.

On the other hand, if we push the reasoning behind Hook's objections one step further, we could wonder why major and minor chords are considered as a single set. Consider for example the usual neo-Riemannian P operation: if one views this operation as an inversion, it is then an involution, i.e $P^2 = 1$ meaning that this operation is formally equal to its inverse. However if one considers this operation as a voice-leading transformation, it then corresponds to :

1. A pitch down in the major-to-minor way.

2. A pitch up in the minor-to-major way.

In this view, the P operation cannot be said to be equal to its inverse. In order to restore coherence in this point of view, one has to consider two different transformations, one from the set of major triads to the set of minor ones, the other one from the set of minor triads to the set of major ones. Notice however we can only do so at the expense of closure: the transformations thus defined only acts on a given set of chords, or in other terms these are partial transformations.

The P operation can thus be viewed as a package of two partial transformations. Notice that this is not a unique case : the L and R operations can

also be viewed as "packaged operators". Moreover, the usual transposition operators in neo-Riemannian theories actually represents two partial transposition operators which apply respectively to the major and minor chords. In Hook's notation of UTT (Uniform Triadic Transformations), these correspond to the two operations $< +, 1, 0 >$ and $< +, 0, 1 >$. Since they act very similarly, it is conceivable to package them into a single transposition operator T_1. The last section of this paper will provide a link between the construction we introduce next and packaged operators.

3 A categorical construction for musical transformations

In this section, we introduce a construction of musical transformations based on categories rather than groups. Notice that groups are themselves a particular case of categories as they can be viewed as single-object categories, where the morphisms are group elements under the usual composition. The construction we will use is based on a generalization of the construction of group extensions that was introduced in [12].

Recall first that the construction of generalized neo-Riemannian groups of transformations as extensions :

$$1 \to Z \to G \to H \to 1$$

involves a base-group Z and a shape group H. The base-group is commonly associated with transpositions operators, i.e $Z = \mathbb{Z}_{12}$ for example, where as the shape group is associated which switches between different set-classes, i.e $Z = \mathbb{Z}_2$ in usual neo-Riemannian theory.

Instead of considering H as a group, we now replace it with a groupoid $\mathcal{H}$. Recall that a groupoid is a category in which every morphism is invertible. Groupoids can be viewed as generalizations of groups in which closure (or totality) has been left out. Indeed, morphisms of the groupoid can be seen as partial transformations between objects. In the rest of the paper, the maps s and t will refer to the source and target maps, i.e if x_{ij} indicates a morphism from object (or set-class) i to j, then $s(x_{ij}) = i$ and $t(x_{ij}) = j$. It has been suggested that groupoids are in some cases superior to groups in describing symmetries of objects. For a gentle introduction to groupoids, the reader is invited to refer to [20] and [21].

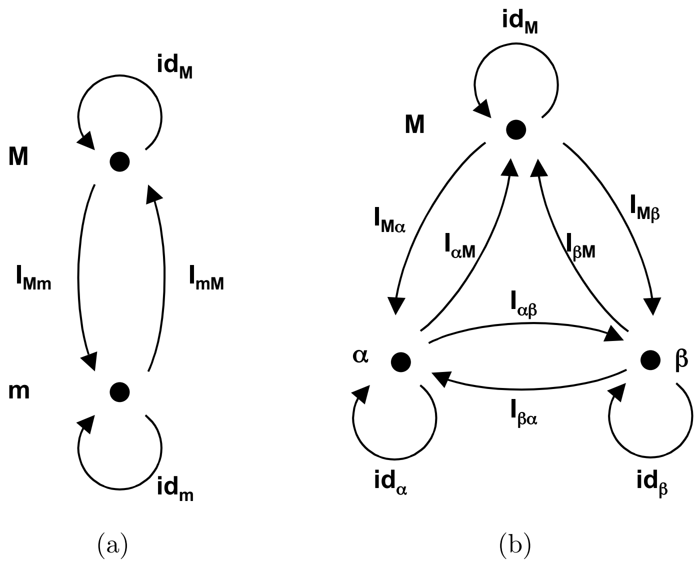


Figure 4: Two examples for categories $\mathcal{H}$ of formal transformations between major and minor chords (a) and major, α, and β chords (b)

The objects of the groupoid $\mathcal{H}$ are the different set-classes and the morphisms are the different formal transformations between these set-classes. By formal transformation we mean an abstract operation allowing the transformation of one set-class into another, in a similar way the group H is considered in [12]. Actual transformations of chords usually involve transpositions of the root as is the case for the L or R operation, which we have not added yet. By definition, these transformations are partial and the composition of two morphisms is only possible if the codomain of the first matches the domain of the second. Figure 4 give two examples of such groupoids, corresponding respectively to transformations between major (M) and minor (m) chords, and M, α, and β chords. R

We now introduce the definition of a category extension, following the work of Hoff [22, 23, 24] :

Definition *(Hoff) An extension of the category $\mathcal{Z}$ by the category $\mathcal{H}$ is a category $\mathcal{G}$ such that there exists a sequence*

$$1 \to \mathcal{Z} \xrightarrow{i} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \to 1$$

in which :

1. *$\mathcal{Z}$,$\mathcal{G}$ and $\mathcal{H}$ have the same number of objects.*

2. *i is a functor injective on morphisms, while π is a functor surjective on morphisms*

3. *$\forall g, h \in \mathcal{G}, \pi(g) = \pi(h) \Leftrightarrow \exists! m \in \mathcal{Z}, h = i(m) \cdot g$*

This definition closely follows the group extension one, and the third condition is actually similar to the $Im(i) = Ker(\pi)$ condition. Hoff has shown that if $\mathcal{H}$ is a category extension as defined above, then $\mathcal{Z}$ is a disjoint union of groups indexed by the objects of $\mathcal{H}$. The category $\mathcal{Z}$ thus plays the role of transposition operators. If we assume that each set-class can be transposed in the same way, for example that there is a simply transitive group action of $\mathbb{Z}_n$ on each set-class, we can therefore consider that $\mathcal{Z}$ is built as such :

1. The objects of $\mathcal{Z}$ are the same as in $\mathcal{H}$ and represent the set-classes.

2. Each object is equipped with copies of all the morphisms of $\mathbb{Z}_n$ viewed as a single-object category. We denote those morphisms as z_{ii} when they belong to object i. These morphisms represent transpositions of the individual set-classes.

3. The set of morphisms between two different objects is empty.

With this knowledge, we see that the third condition in the definition of a category extension has a very concrete meaning from a musical point of view. Consider for example the L and R operations acting on the C major triad. It is clear that the images of C_M under L and R differ by a unique transposition. We thus axiomatize this fact by considering a construction, as a category extension, in which any two switching transformations differ only by a unique transposition in the target set-class.

The role of the functor i is to introduce the transposition operators in the category $\mathcal{H}$. To sum up, the functors i and π are defined as :

1. i and π maps objects in the natural way.

2. i maps morphisms z_{ii} of $\mathcal{Z}$ to equivalent transposition morphisms z_{ii} in $\mathcal{G}$. By an abuse of terminology, z_{ii} will designate from now on transposition morphisms from $\mathcal{Z}$ or from $\mathcal{G}$ indifferently.

3. π maps morphisms z_{ii} in $\mathcal{G}$ to id_i in $\mathcal{H}$, and morphisms x_{ij} in $\mathcal{G}$ to morphisms x_{ij} in $\mathcal{H}$.

We then have some obvious propositions characterizing some basic structural facts about $\mathcal{G}$:

Proposition *If $\mathcal{G}$ is a extension of $\mathcal{Z}$ by $\mathcal{H}$ and $\mathcal{H}$ is a groupoid, then $\mathcal{G}$ is a groupoid.*

Proof Let g be a morphism of $\mathcal{G}$. If g is a transposition morphism z_{ii}, then it obviously has an inverse. If g is an inter-object morphism g_{ij}, observe first that there is always at least one morphism x_{ji} in $\mathcal{G}$ $\forall i, j$. We thus have :

$$x_{ji} \circ g_{ij} = z_{ii}^k$$

and since z_{ii}^k is inversible, $z_{ii}^{-k} \circ x_{ji}$ is the inverse of g_{ij}.

$\square$

Proposition *If $\mathcal{H}$ is a groupoid, $\mathcal{Z}$ a disjoint union of finite cyclic groups $\mathbb{Z}_n$, and $\mathcal{G}$ is a full extension of $\mathcal{Z}$ by $\mathcal{H}$ then :*

1. $z_{ii}^n = id_i, \forall i$
2. $\forall (x_{ij}, x_{ji}), \exists p = p(x_{ij}, x_{ji}) \in [1..n], x_{ji} \circ x_{ij} = z_{ii}^p$
3. $\forall (x_{ij}, x_{ji}), \exists q = q(x_{ij}, x_{ji}) \in [1..n], x_{ji} \circ z_{jj} \circ x_{ij} = z_{ii}^q$.
4. $\forall (x_{ij}, x'_{ij}), \exists r = r(x_{ij}, x'_{ij}) \in [1..n], x_{ij} = x_{jj}^r \circ x'_{ij}$.

Proof The propositions are trivial by definition of a category and by the extension structure.

$\square$

These relations generalize the presentation of extensions of cyclic groups $\mathbb{Z}_n$ by cyclic groups $\mathbb{Z}_2$:

$$G = \langle z, x | z^n, x^2 = z^p, x^{-1} z x = z^q \rangle$$

these metacyclic groups being found in particular in generalized neo-Riemannian groups (see [12]).

The extension construction of $\mathcal{G}$ brings however more structure with regards to morphism composition, and allows to define actions of $\mathcal{G}$ on sets of objects. Indeed, Hoff has shown that, in the case the groups of $\mathcal{Z}$ are abelian, all category extensions $1 \to \mathcal{Z} \to \mathcal{G} \to \mathcal{H} \to 1$ can be constructed as such :

1. $\mathcal{G}$ has the objects of $\mathcal{H}$ or $\mathcal{Z}$.

2. Morphisms of $\mathcal{G}$ are of the form (z, h), i.e they are indexed by the morphisms from $\mathcal{H}$ or $\mathcal{Z}$, with z being a morphism from $t(h)$ to $t(h)$.

3. Composition of morphisms, whenever they are compatible (i.e $s(h_2) = t(h_1)$) is given by the law :

$$(z_2, h_2) \cdot (z_1, h_1) = (z_2 \circ \phi_{h_2}(z_1) \circ \zeta(h_2, h_1), h_2 \circ h_1)$$

   where ϕ is an action of the category $\mathcal{H}$ on $\mathcal{Z}$, and ζ is a 2-cocycle.

The cohomology theory built by Hoff allows to classify all category extensions based on the second cohomology group $H^2(\mathcal{H}, \mathcal{Z})$, with the corresponding 1- and 2-cocycles. We now give the definitions for the terms involved in the composition law of morphism.

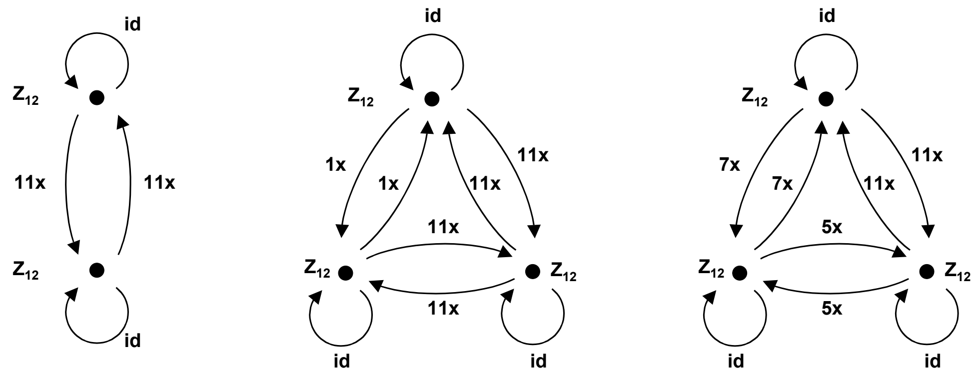
An action ϕ of $\mathcal{H}$ on $\mathcal{Z}$ is a functor $\phi : \mathcal{H} \to \mathbf{Grp}$ where the images of the objects of $\mathcal{H}$ are the groups associated to the corresponding objects in $\mathcal{Z}$. In other terms, this functor defines homomorphisms between the groups of $\mathcal{Z}$ which are compatible with composition of functors in $\mathcal{H}$.

Some examples of actions in the case of M and m set-classes, or M, α and β chords are given in Figure 5. Figure 5(a) shows the typical homomorphisms which are used in neo-Riemannian theory. Figure 5(b) and Figure 5(c) show new structures: in particular one can easily recognize the homomorphisms of 5(b) in the partial transformations we introduced above. Figure 5(c) show a new possibility for combining homomorphisms. The term $\phi_{h_2}(z_1)$ in the composition law of morphisms corresponds therefore to a transfer of the transposition operator from a set-class into another.

A 2-cocycle is a function $\zeta : \mathcal{H} \times \mathcal{H} \to \mathcal{Z}$ between two morphisms of $\mathcal{H}$ which outputs a morphism from the appropriate object of $\mathcal{Z}$ such that :

$$\phi_{h_3}(\zeta(h_2, h_1)) \circ \zeta(h_3, h_2 \circ h_1) = \zeta(h_3, h_2) \circ \zeta(h_3 \circ h_2, h_1)$$

whenever h_1, h_2 and h_3 are compatible.



(a) (b) (c)

Figure 5: Some examples of actions of $\mathcal{H}$ on $\mathcal{Z}$ for set classes M and m, or M, α and β. We show here the images of the functor $\phi : \mathcal{H} \to \mathbf{Grp}$. The homomorphisms between groups are represented by their multiplicative action.

We see that the terminology used for category extensions is very close to the one used for group extensions. In a similar approach, we now define the actions of $\mathcal{G}$.

A left- (right-) action of $\mathcal{G}$ on a set of chords can be described as covariant (contravariant) functor $F : \mathcal{G} \to \mathbf{Set}$. If one needs an analog of simply transitive actions of groups on sets (i.e a G-torsor), one then needs to consider representable functors $F : \mathcal{G} \to \mathbf{Set}$, i.e functors which are naturally isomorphic to $Hom(X, -)$ for some object X of $\mathcal{G}$ (since $\mathcal{G}$ is a groupoid, the choice of the object is not important). In the case of groups, it can be shown that $F : G \to \mathbf{Set}$ is representable if and only if G has a simply transitive action on the corresponding set. Recall that in that case, set elements can be put in bijection with group elements after a particular element is identified to the identity element in the group. As shown in [12], this allows the determination of group actions. By analogy, given a representable functor $F : \mathcal{G} \to \mathbf{Set}$ and using Yoneda's lemma, we can put morphisms in $\mathcal{G}$ and set elements in bijection by choosing one particular element and identifying it with the identity element in the corresponding set-class. Explicit actions can then be calculated from the morphisms composition law.

We see here that considering groupoids and their extensions allow for much richer structure than the group extension structure does. In partic-

ular, the interplay of group homomorphisms between set-classes, as shown in Figure 5 is a way to circumvent the limitations of group extensions when considering the automorphisms of the only group $\mathbb{Z}_{12}$.

4 Forming groups of transformations from category extensions

Starting from a groupoid $\mathcal{G}$ of musical transformations defined as a full extension, it is possible to revert back to a group-theoretical description by "packaging" partial operations.

Definition *A packaged operator is a set of morphisms $O = \{\phi_1, ..., \phi_n\}$ from $\mathcal{G}$ (n being the number of objects of $\mathcal{G}$) such that for all objects i, i appear only once as the domain of a morphism from O, and only once as the codomain of a morphism from O.*

Packaged operators can be composed according to :

Definition *The composition $O_1 \circ O_2$ of two packaged operators $O_1 = \{\phi_1, ..., \phi_n\}$, $O_2 = \{\phi'_1, ..., \phi'_n\}$ is the set of morphisms $\{\phi''_1, ..., \phi''_n\}$ obtained by composing all morphisms $\phi_x \circ \phi'_y$ from O_1 and O_2 whenever their domain and codomain are compatible. It can be verified that $O_1 \circ O_2$ is also a packaged operator.*

We then have :

Proposition *Packaged operators form a group under composition.*

Proof The identity packaged operator is the set of identity morphisms of each object. Closure is given by definition. Associativity is inherited from the category structure. Finally, since $\mathcal{H}$ is a groupoid it is always possible to find inverses for each morphism of a packaged operator, thus giving the inverse packaged operator.

$\square$

For example, one can define a packaged transposition operator given by the set of generators z_{ii} of $\mathbb{Z}_n$ for each object i of $\mathcal{H}$, i.e the transposition operator for each set-class. This packaged transposition operator has the form $\{z_{ii}, z_{jj}, z_{kk}, ...\}$ and naturally generates the cyclic group $\mathbb{Z}_n$. One can also consider packaged operators of the form $\{z_{ii}, id_j\}, \forall j \neq i$, for all objects i of $\mathcal{G}$. The set of all these transposition operators then form a direct product of copies of $\mathbb{Z}_n$.

In the same view, one can package inversion morphisms I_{Mm}, I_{mM} in a single operator, thus recovering the usual neo-Riemannian inversion transformation.

In the case of the three set-classes M, α and β, one could consider on one hand a packaged inversion operator $I = \{\phi_{M\alpha}, \phi_{\alpha\beta}, \phi_{\beta M}\}$, or on the other hand three different packaged inversion operators $I_1 = \{\phi_{M\alpha}, \phi_{M\alpha}, id_\beta\}$,$I_2 = \{id_M, \phi_{\alpha\beta}, \phi_{\beta\alpha}\}$ and $I_3 = \{\phi_{M\beta}, id_\alpha, \phi_{\beta M}\}$.

The next proposition makes the link between such packaged operators and the wreath products that appeared in the work of Hook and Peck. Let $\mathcal{H}$ be a groupoid, $\mathcal{Z}$ a disjoint union of m finite cyclic groups $\mathbb{Z}_n$, and $\mathcal{G}$ an extension of $\mathcal{Z}$ by $\mathcal{H}$. Consider one hand the set of all packaged transposition operators $N = \{T_i\}$, where $T_i = \{z_{ii}, id_j\}, j \neq i, (i,j) \in \{1...m\}$, assuming that z_{ii} is a generator of $\mathbb{Z}_n$. Consider on the other hand the set of all packaged inversion operators $K = \{I_{ij}\}$, where $I_{ij} = \{x_{ij}, x_{ji}, id_k\}, k \neq i \neq j, (i,j,k) \in \{1...m\}$, such that $x_{ij} = x_{ji}^{-1}$.

Obviously, N and K are groups, N being isomorphic to a direct product of m copies of $\mathbb{Z}_n$, while K is isomorphic to the symmetric group S_m (I_{ij} are involutions by definition, and it is easy to verify that $I_{ij} \circ I_{jk} \circ I_{ij} = I_{jk} \circ I_{ij} \circ I_{jk}$). We then have :

Proposition *G=$\langle T_i, I_{ij}\rangle$ is isomorphic to the wreath product $\mathbb{Z}_n \wr S_m$*

Proof We first show that N is normal in G. Let n be an element of N, i.e of the form $\{z_{ii}^p, id_j\}, j \neq i$ for some i. If $(g \in G) \in \{T_i\}$ then it is obvious that $g.n.g^{-1} \in N$. If $(g \in G) \in \{I_{ij}\}$ then

$$I_{ij} \circ n \circ I_{ij}^{-1} = \{x_{ji} \circ x_{ij} \circ z_{ii}^p, x_{ij} \circ x_{ji}, id_k\} = \{z_{ii}^p, id_j, id_k\} \in N$$

If $g \in G$ is a composite element of T_i and I_{ij}, the $g.n.g^{-1} \in N$ relation holds with the previous results.

We then have naturally $NK = G$ and $N \cap K = \{id_1, ..., id_m\}$ thus showing that G is a semidirect product of N by K.

□

The previous definition of G requires incorporating all individual transposition operators for each object. It is possible to include only one, based on the following condition :

Proposition *Let $T_1 = \{z_{11}, id_k\}, k \neq 1$. If $x_{1i} \circ z_{11} \circ x_{i1} = z_{ii}^q$ is a generator of $\mathbb{Z}_n, \forall i$, then G'=$\langle T_1, I_{1i} \rangle$ is isomorphic to G*

Proof We have

$$I_{1i} \circ T_1 \circ I_{1i} = x_{1i} \circ z_{11} \circ x_{i1} = z_{ii}^p$$

which generates a copy of $\mathbb{Z}_n$, $\forall i$. Therefore G' has a subgroup which is isomorphic to N and the rest of the demonstration follows.

□

One can check for example that the packaged partial operations defined on M, α and β chords in section 2.1, along with the packaged transposition operator $\{n_M \to n+1_M, n_\alpha \to n_\alpha, n_\beta \to n_\beta\}$, generate a group of order 10368 which is isomorphic to $\mathbb{Z}_{12} \wr S_3$.

5 Conclusions

We have introduced in this paper a categorical construction for musical transformations based on groupoids which extend the precedent construction based on group extensions. It overcome its inherent limitations, in particular the limited choice of automorphims in the pc-set group. More importantly, this construction allows to define compatible set of partial transformations between pair of set-classes. We also saw how groups of transformations can be recovered from category extensions, based on packaged operators and their composition.

While this paper is more mathematical than musical, we hope it will provide foundations for building appropriate groups of transformations in

musically-relevant domains. This could be applied for example to cardinality changes between chords (ex. major/minor to seventh chords), a very important problem in music theory as of now.

In this paper, we only considered the case of groupoids, and in particular the groupoid $\mathcal{H}$ which assume that partial and reversible transformations between set-classes always exist. As seen in the work of Cohn regarding major/minor and augmented triads, there are cases in music theory where partial transformations may not be reversible at all. It would therefore be interesting to consider category extensions in which $\mathcal{H}$ is a more general category. As well, it could also be interesting to investigate non-abelian category extensions, i.e in which the groups of $\mathcal{Z}$ are non-abelian.

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